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# Evolutionary dynamics for economic behaviour: competition versus optimization

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**Abstract.** In an earlier paper the authors applied evolutionary dynamics to describe the survival of coalitionist versus non-coalitionist behaviour in a bi-matrix game. Now, in a dynamic situation the survival of "competitor" versus "optimizer" behaviour will be studied.

In classical game theory the action of a player (called optimizer) is to optimize his own pay-off. In evolutionary game theory instead, a Darwinian competitor always wants to be better than the average.

In this paper, starting out from a classical symmetric two-person normal form game, a dynamic set up is developed in which it can be predicted whether the competitor or the optimizer will survive in the long run.

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## 1 Introduction

The aim of this paper is to develop a dynamic model in which it can be predicted whether in a game-theoretical conflict the competitor or the optimizer behaviour will survive in the long run.

In classical game theory the concepts of solution very much depend on the aim of the player, which in particular means that each player makes every effort to optimize his own pay-off.

In Darwinian "struggle for existence" the main point is that an individual must be better than its competitor. For instance, Hamilton (1970, 1971) pointed out that a "spiteful trait" (an animal is spiteful if it harms both itself and another) will be selected if the spiteful individual decreases his own fitness less than that of its partner. We emphasize that here for the survival of the spiteful the strong interaction is a prerequisite.

It is known that in the selection situation not always profit maximizing is the best survival strategy, in economy either. As we know, first Hansen and

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Samuelson (1988) pointed out that if the agent has negligible effect on others' pay-off then the pay-off maximizer will survive, but when this assumption is not met, survival need not imply optimality. Later, based on Hamilton's theory, Schaffer (1989) demonstrated that if the firms have market power (i.e. the firm's success depends on other players' behaviour, cf. Güth and Peleg 2001) then the profit-maximizers are not the best survivors. The reason is that although the 'spiteful' firm decreases its pay-off as well, it decreases the maximizer's pay-off even more. Schaffer's idea is similar to ours, but there is a major difference: he applied the standard biological set up, namely he considered a "population" of firms that never change their strategies, and the selection process guarantees the long-term realization of a Nash equilibrium. We will consider rational players, who can change their strategies in a smooth way according to their pay-off functions and have complete information about the current state of the game. Moreover, our approach is dynamical, since in it both the behaviour of the players and the selection process is described by differential equations.

As we have seen above, the main difference between the attitudes in biology and in economy is that in biology the success depends on the relative advantage of the individual (i.e. on the difference between the pay-off of the individual and the average pay-off of the whole population), while in economy success means higher pay-off only. The basic question is, which behaviour can survive eliminating the other one, or the coexistence of both behaviour types is also possible? In this paper we set up a minimal model in which we can deal with this problem.

The paper is organized as follows.

In Section 2 we set up a general selection model in which we can deal with the different behaviour types. Our approach consists in two main steps.

*First*, we introduce a new, modified game where players are the following: Player 1 is the optimizer, and Player 2 is the competitor. The strategy sets of new game are the same as those of the starting game. We introduce the "weight of a player" (between 0 and 1) expressing his relative importance or presence in the game (player having zero weight will be eliminated from the game). The new pay-off of each player will be defined as a modification of the original pay-off, which depends both on the strategy choices of the players and on the weight of the considered player. The definition of new the pay-off functions strongly depends on the behaviour type of the player (cf. Bester and Güth, 1998, where a model for the altruist behaviour is built up.)

*Second*, we introduce a dynamical approach, which is based on the following assumptions: Each player is supposed to change his strategy in function of time, according to his behaviour as competitor or optimizer. Moreover, we assume that each player can change his actual strategy in a continuous way, corresponding to the direction of the gradient of his pay-off function. Finally, we suppose that the change of weight of the players is described by the replicator dynamics. This replicator dynamics determines which behaviour type will survive. This dynamics, however, is based on the income rather than the pay-off function of the modified game.

In Section 3 an illustrative example is given, where the pay-off of the game

between competitor and optimizer is quadratic. We will show that there is no totally mixed equilibrium, which means that the two behaviour types can not coexists. Moreover, for the case of two pure strategies, we obtain that the exclusive presence of the competitor is a locally asymptotically stable equilibrium state, which means that the competitor can eliminate an emerging optimizer.

# 2 Dynamical model for the selection of different behaviour types

For the sake simplicity we consider pair-wise conflicts. Let us start out from a classical symmetric two-person game with strategy set $S \subset \mathbb{R}$ for both players, and pay-off function $\varphi : S \times S \to \mathbb{R}$ for the first player (for strategy choices $(p, q)$ the pay-off of the second player is $\varphi(q, p)$). Symmetry is supposed because later on we want to study only the neat and clean effect of the difference in the behaviour types, without being covered by other differences among players. To set up our model for the selection of different behaviour types we will follow the plan sketched in the introduction.

## 2.1 The modified game

Let us introduce a modified game, where two different behaviour types are considered players: competitor and optimizer. Players differ in their behaviour types (attitudes): there are two players, and they have the same strategy set as the starting game. Denote by $p, q \in \mathbb{R}$ the strategies of competitor (Player 1) and optimizer (Player 2), respectively. For the description of the selection of players we introduce the *weight* of Player 1, $x \in [0, 1]$, and $1 - x$ is interpreted as the weight of Player 2. (The weight $x$ may express, for instance, the market share of Player 1.) A player with zero weight is considered eliminated.

Of course, the introduction of the weight of the players brings a new structure into the game, and in concrete cases the new, strategy- and weight-dependent "income" function can be constructed from the original pay-off $\varphi$ (see Example 1 below). In general terms, for $i \in \overline{1, 2}$ let $\psi_i : S \times S \times [0, 1] \to \mathbb{R}$ be the *income function* of player $i$.

Let us now define the pay-off functions in the modified game, corresponding to the different behaviour types in the following way.

Assume that the *competitor* wants to maximize his relative advantage over the average.

$$\psi_1(p, q, x) - [x\psi_1(p, q, x) + (1 - x)\psi_2(p, q, x)] = (1 - x)\left[\psi_1(p, q, x) - \psi_2(p, q, x)\right].$$

Since for fixed $q$ and $x$, the competitor can increase his relative advantage by the choice of his strategy $p$ only if in terms of income he is better than the other player, increasing the difference

$$K(q, p, x) := \psi_1(p, q, x) - \psi_2(p, q, x), \tag{1}$$

called simply *advantage* of the competitor. Therefore, $K$ will be considered the *pay-off function of the competitor*. We emphasize again that in the modelled situation, competitor wants to be better than the optimizer.
The *optimizer* wants to maximize its own income, i.e. his *pay-off* is $\psi_2(p,q,x)$.

Summing up, for the modified game we have $S$ as strategy set for both players, $[0,1]$ as set of weights, pay-offs $K$ and $\psi_2$, for the competitor and the optimizer, respectively.

EXAMPLE 1 For $(q,p,x) \in \mathbb{R} \times \mathbb{R} \times [0,1]$ let $\psi_1(p,q,x) := x\varphi(p,p) + (1-x) \times\varphi(p,q)$ and $\psi_2(q,p,x) := x\varphi(q,p) + (1-x)\varphi(q,q)$. Then the pay-off function of the competitor type is $K(q,p,x) := x\left(\varphi(p,p) - \varphi(p,q)\right) + (1-x)\left(\varphi(q,p) - \varphi(q,q)\right)$.

## 2.2 Dynamics for the strategies and the weight

For our dynamic model let us assume that the strategy choice of the players in the modified game is restricted in the sense that they can only change their strategies in a "smooth" way and they do this efficiently, as to achieve a maximum increase of their respective pay-offs. Therefore, we shall suppose that the *strategy changes* are described by the so-called *partial gradient dynamics* (see e.g. Garay, 2002). In our simple case when $S := \mathbb{R}$, the dynamics for the competitor and the optimizer behaviour types, respectively will be the following:

$$\begin{aligned} \dot{p} &= \frac{\partial\left[\psi_1(p,q,x) - \psi_2(p,q,x)\right]}{\partial p} \\ \dot{q} &= \frac{\partial\psi_2(p,q,x)}{\partial q}. \end{aligned} \tag{2}$$

REMARK 1 For the common strategy set we chose $\mathbb{R}$ in order to reduce technical complications. If we take $S := S_n \subset \mathbb{R}^n$, the standard simplex must be positively invariant for the considered dynamics. Then the corresponding *partial gradient dynamics* would be the following:

*for player 1*

$$\dot{p}_i = p_i(1-p_i)\frac{\partial\left[\psi_1(p,q,x) - \psi_2(p,q,x)\right]}{\partial p_i},$$

$$\dot{p}_n = 1 - \sum_{i=1}^{n-1} \dot{p}_i;$$

*for player 2*

$$\dot{q}_j = q_j(1-q_j)\frac{\partial\psi_2(p,q,x)}{\partial q_j},$$

$$\dot{q}_n = 1 - \sum_{j=1}^{n-1} \dot{q}_j;$$

guaranteeing the required invariance.

For the change of *the weight of players* we will suppose that it can be described by the well-known *replicator dynamics* which in evolutionary game theory determine the long-term outcome of the evolution of behaviour phenotypes (see e.g. Hofbauer & Sigmund 1999). Hence for the weight $x$ of the competitor we have

$$\begin{aligned}\dot{x} &= x\left(\psi_1(p,q,x) - [x\psi_1(p,q,x) + (1-x)\psi_2(p,q,x)]\right)\\ &= x(1-x)\left(\psi_1(p,q,x) - \psi_2(p,q,x)\right). \end{aligned} \tag{3}$$

Intuitively, if the income of the competitor is greater than the average, its weight will increase, and decrease in the opposite case.

# 3 An illustrative example

In this section we give a first illustration of the possible tools for the dynamical analysis of the model of the previous section. At this stage of the research the motivation for the choice of the functions $\psi_1$ and $\psi_2$ is merely technical, we wanted to deal with linear dependence on the weight and a quadratic dependence on the strategies, which is general enough to avoid degeneration in the calculations.

With given parameters $a, b, c, d \in \mathbb{R}$ with $a \neq 0$, for $(p, q, x) \in \mathbb{R} \times \mathbb{R} \times [0, 1]$ we define

$$\begin{aligned}\psi_1(p,q,x) &:= x(p-q)(ap+b) + p(aq+c) + bq + d\\ \psi_2(p,q,x) &:= x(p-q)(aq+b) + q(aq+c) + bq + d.\end{aligned}$$

We notice that the general form of the above income functions $\psi_1, \psi_2$ is motivated by the following biological (and also social) situation: Let us suppose that $p$ and $q$ are the respective strategies displayed by a competitor and optimizer individual in a random pair-wise conflict described by a $2 \times 2$ matrix game. Let $x$ and $1 - x$ be the frequency of the competitor and optimizer behaviour types, respectively. Then the average income of an individual of the respective behaviour types are $\psi_1$ and $\psi_2$.

Now, from equations (2)–(3) for the respective strategies $p$ and $q$ of competitor and optimizer, and for the weight $x$ of the optimizer we have the following dynamics:

$$\dot{p} = 2xa(p-q) + aq + c \tag{4}$$

$$\dot{q} = 2a(1-x)q - bx + axp + c + b \tag{5}$$

$$\dot{x} = x(1-x)(p-q)\left[ax(p-q) + aq + c\right]. \tag{6}$$

From the particular structure of the right-hand side of equation (6), it easily follows that the set $\mathbb{R} \times \mathbb{R} \times [0, 1]$ is positively invariant for system (4)–(6).

First, we show that system (4)–(6) has no interior equilibrium. For $x \in ]0, 1[$,

the equations for the equilibrium are

$$2xa(p-q)+aq+c=0 \tag{7}$$
$$-2axq-bx+axp+2aq+c+b=0 \tag{8}$$
$$(p-q)\left[ax(p-q)+aq+c\right]=0. \tag{9}$$

If in (9) we have $(p-q)=0$, from (7) we obtain $p=q=-\dfrac{c}{a}$, and equation (8) reduces to

$$-ax\left(-\frac{c}{a}\right)-bx+2a\left(-\frac{c}{a}\right)+c+b=0,$$

implying $x=-1$, in contradiction to $x\in]01[$. Alternatively, suppose that in equation (9) the second factor is zero. Then, by (4) we get $2ax(p-q)=0$, implying again $p=q$ and hence $x=-1$, too. Therefore, in both cases, the existence of an interior equilibrium is excluded.

For non-interior equilibria let us consider two cases.

*Case 1.* From equations (7) and (8) it is easy to see that equilibrium with $x:=0$ is possible only if $b=c$, and in this case $(-c/a,p,0)$ is an equilibrium for any $p\in\mathbb{R}$. This degenerate case can be avoided supposing $b\neq c$.

*Case 2.* Now we look for an equilibrium with $x=1$. Then

$$\begin{aligned}2ap-aq+c&=0\\ ap+c&=0.\end{aligned}$$

Hence the solution is $\bar{p}:=-\dfrac{c}{a}$, $\bar{q}:=-\dfrac{c}{a}$, and the corresponding equilibrium is $(\bar{p},\bar{p},1)$.

For a stability analysis we linearize system (4)–(6) at equilibrium $(\bar{p},\bar{p},1)$. The Jacobian of the right-hand side at $(\bar{p},\bar{p},1)$ is

$$C:=\begin{pmatrix}2a & -a & 0\\ a & 0 & c-b\\ 0 & 0 & 0\end{pmatrix},$$

with characteristic polynomial

$$\lambda(\lambda^2-2a\lambda+a^2).$$

Hence $C$ has eigenvalues $\lambda_{1,2}=a$ and $\lambda_3=0$. Therefore, for a stability analysis we shall need the *centre manifold theory* (see Appendix). From now on let us suppose that $b\neq c$.

First we translate the equilibrium $(\bar{p},\bar{p},1)$ into the origin by introducing $\xi:=p-\bar{p}$, $\eta:=q-\bar{p}$, $\zeta:=x-1$. Then system (4)–(6) takes the form

$$\dot{\xi}=2x\xi-a\eta+2a\zeta(\xi-\eta) \tag{10}$$
$$\dot{\eta}=2a\xi(c-b)\zeta-\zeta a(2\eta-\xi) \tag{11}$$
$$\dot{\zeta}=-a\zeta(1+\zeta)(\xi-\eta)\left[\xi+\zeta(\xi-\eta)\right]. \tag{12}$$

For the application of the centre manifold technique we transform the linear part to block-diagonal form using the transformation matrix

$$U := \begin{pmatrix} 1 & 0 & \dfrac{c-b}{a} \\ 0 & 1 & 2\dfrac{c-b}{a} \\ 0 & 0 & \dfrac{b-c}{a} \end{pmatrix}$$

$$r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} := U \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} \xi + \dfrac{c-b}{a}\zeta \\ \eta + 2\dfrac{c-b}{a}\zeta \\ \dfrac{b-c}{a}\zeta \end{pmatrix}.$$

Now the transformed system reads as follows:

$$\begin{aligned} \dot{r}_1 &= 2ar_1 - ar_2 + 2\frac{a^2}{b-c}r_3(r_1 - r_2 - r_3) \\ &\quad + ar_3\left(1 + \frac{a}{b-c}r_3\right)(r_1 - r_2 - r_3)\left[r_1 + r_2 + \frac{a}{b-c}r_3(r_1 - r_2 - r_3)\right] \end{aligned} \tag{13}$$

$$\begin{aligned} \dot{r}_2 &= ar_1 - 2\frac{a^2}{b-c}r_3(r_1 - r_2 - r_3) \\ &\quad + 2ar_3\left(1 + \frac{a}{b-c}r_3\right)(r_1 - r_2 - r_3)\left[r_1 + r_2 + \frac{a}{b-c}r_3(r_1 - r_2 - r_3)\right] \end{aligned} \tag{14}$$

$$\dot{r}_3 = -ar_3\left(1 + \frac{a}{b-c}r_3\right)(r_1 - r_2 - r_3)\left(r_1 + r_2 + \frac{a}{b-c}r_3(r_1 - r_2 - r_3)\right). \tag{15}$$

The Jacobi matrix of the right-hand side at zero is

$$\begin{pmatrix} 2a & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the nonlinear parts of (13)–(14), together with their respective derivatives vanish at the origin.

The basic existence theorem of centre manifold theory (see Theorem A1 of the Appendix) guarantees the existence of a centre manifold for system (13)–(15). Let us find the centre manifold in the form

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} h_1(r_3) \\ h_2(r_3) \end{pmatrix} = h(r_3).$$

For functions $H : \mathbb{R} \to \mathbb{R}^2$ which are continuously differentiable in a neighbourhood of the origin we define the operator

$$(MH)_1(r_3) := H_1'(r_3)\dot{r}_3 - \dot{r}_1$$
$$(MH)_2(r_3) := H_1'(r_3)\dot{r}_3 - \dot{r}_2.$$

Considering $H$ as an approximation of $h$, let us suppose that $H(r_3) = O(r_3^2)$ as $r_3 \to 0$. It is easy to see that $\dot{r}_3 = O(r_3^3)$, and hence

$$(MH)_1(r_3) = -\left(2aH_1(r_3) - aH_2(r_3) - 2\frac{a^2}{b-c}r_3^2\right) + O(r_3^3) = O(r_3^2)$$
$$(MH)_2(r_3) = -\left(aH_1(r_3) - 4\frac{a^2}{b-c}r_3^2\right) + O(r_3^3) = O(r_3^2).$$

Since $H(r_3) = O(r_3^2) = o(r_3)$, we clearly have $H(0) = 0$ and $H'(0) = 0$. Now we choose a function $H$ such that the terms in parenthesis are equal to zero and apply Theorem A3 of the Appendix on the approximation of the centre manifold. From

$$H_1(r_3) = 4\frac{a}{b-c}r_3^2$$

we obtain

$$h_1(r_3) = 4\frac{a}{b-c}r_3^2 + O(r_3^3),$$

and from

$$2aH_1(r_3) - aH_2(r_3) - 2\frac{a^2}{b-c}r_3^2 = 0$$

we get

$$H_2(r_3) = 6\frac{a}{b-c}r_3^2$$

implying

$$h_1(r_3) = 6\frac{a}{b-c}r_3^2 + O(r_3^3).$$

Now we apply Theorem A2 of the Appendix for the (local) stability analysis of system (13)–(15) at the origin. This theorem says that to this end it is enough to see the stability properties of the trivial equilibrium of the equation obtained from (15) substituting $r_1 = h_1(r_3)$ and $r_2 = h_2(r_3)$. A simple calculation shows that

$$h_1(r_3) - h_2(r_3) - r_3 = -r_3\left[2\frac{a}{b-c}r_3 + 1 + O(r_3^2)\right],$$
$$r_1 + r_3 = -r_3 + 1 + O(r_3^2),$$

and as a result

$$\begin{aligned}\dot{r}_3 &= ar_3^3\left(1 + \frac{a}{b-c}r_3\right)\left(2\frac{a}{b-c}r_3 + 1 + O(r_3^2)\right)\\ &\quad\times\left(3\frac{a}{b-c}r_3 + 1 + O(r_3^2)\right)\\ &= ar_3^3 + o(r_3^5).\end{aligned}$$

Since $a < 0$, zero is (locally) asymptotically stable, implying (local) asymptotic stability of equilibrium $(\bar{p}, \bar{p}, 1)$.

Summing up, for our illustrative model we obtained the following result:

THEOREM 1 *If $b \neq c$, then the only equilibrium with $x := 1$ is (locally) asymptotically stable for $a < 0$ and unstable for $a > 0$.*

An intuitive explanation to unstability for $a > 0$ is that in this case, checking the effect of the interaction of two individuals using strategies near $\bar{p}$ we find that it is positive (i.e. cooperative). Therefore the interaction results in the increase of the weight $(1 - x)$ of the optimizer.

# 4 Conclusion

In this paper we have developed a dynamical model in which the survival of the different behaviour types can be investigated. In our set-up we have demonstrated that the competitor and optimizer can not coexist and, under simple conditions on the model parameters, an optimizer with sufficiently small weight will be selected out by the competitor. (In other words, the equilibrium state corresponding to the exclusive presence of the competitors is locally asymptotically stable). If, unlike our case, the strategy sets of the players are compact and the model parameter $a$ is positive, then the two different strategies may coexist in the long run.

We hope that an extended version of our model setup may be suitable for the investigating of different occurring behaviour types of actors of economy, as well.

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# Appendix

From Carr (1981) we recall some definitions and theorems concerning the centre manifold theory. For given matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ with $\operatorname{Re}\lambda = 0$ for all $\lambda \in \sigma(A)$, $\operatorname{Re}\mu < 0$ for all $\mu \in \sigma(B)$ and twice continuously differentiable functions $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, let us consider system

$$\dot{x} = Ax + f(x, y) \tag{A1}$$

$$\dot{y} = By + g(x, y) \tag{A2}$$

with equilibrium $(0, 0)$, i.e. $f(0, 0) = 0$ and $g(0, 0) = 0$. We also suppose that for the Jacobians we have $f'(0, 0) = 0$ and $g'(0, 0) = 0$.

Definition A1 Let $N$ be an invariant manifold for system (A1)–(A2) described by equation $y = h(x)$ where $h : V \to \mathbb{R}^m$ is a twice continuously differentiable function defined in a neighbourhood $V$ of 0. If $h(0) = 0$ and $h'(0) = 0$, then $N$ is called a centre manifold for system (A1)–(A2).

Theorem A1 *There exists a centre manifold for system* (A1)–(A2).

System (A1)–(A2) restricted to the centre manifold can be written in the form

$$\dot{u} = Au + f(u, h(u)). \tag{A3}$$

The next theorem says that system (A3) contains all the necessary information to determine the asymptotic behaviour of the solutions of system (A1)–(A2) near the equilibrium.

THEOREM A2

a) *Suppose that the zero solution of* (A3) *is stable (asymptotically stable, unstable). Then the zero solution of* (A1)–(A2) *is stable (asymptotically stable, unstable).*

b) *Suppose that the zero solution of* (A3) *is stable, and let* $(x, y)$ *be a solution of* (A1)–(A2) *initially sufficiently small. Then there exists a solution* $u$ *of* (A3) *such that with an appropriate* $\gamma > 0$, *as* $t \to +\infty$ *we have*

$$x(t) = u(t) + O(e^{-\gamma t}) \tag{A4}$$

$$y(t) = h(u(t)) + O(e^{-\gamma t}). \tag{A5}$$

Now for any continuously differentiable function $H$ mapping a neighbourhood of the origin of $\mathbb{R}^n$ into $\mathbb{R}^m$, define the mapping $M$ in the following way:

$$(MH)(x) := H'(x)[Ax + f(x, H(x))] - BH(x) - g(x, H(x)).$$

An easy calculation shows that for any $x$ from the domain of $h$ we have $(Mh)(x) = 0$.

The following theorem allows us to approximately calculate the centre manifold.

THEOREM A3 *Let be a continuously differentiable function mapping a neighbourhood of the origin of* $\mathbb{R}^n$ *into* $\mathbb{R}^m$, *satisfying conditions with* $H(0) = 0$ *and* $H'(0) = 0$. *Assume that with some* $q > 1$ *for* $x \to 0$, $(MH)(x) = O(|x|^q)$ *holds. Then for* $x \to 0$ *we have* $|h(x) - H(x)| = O(|x|^q)$.