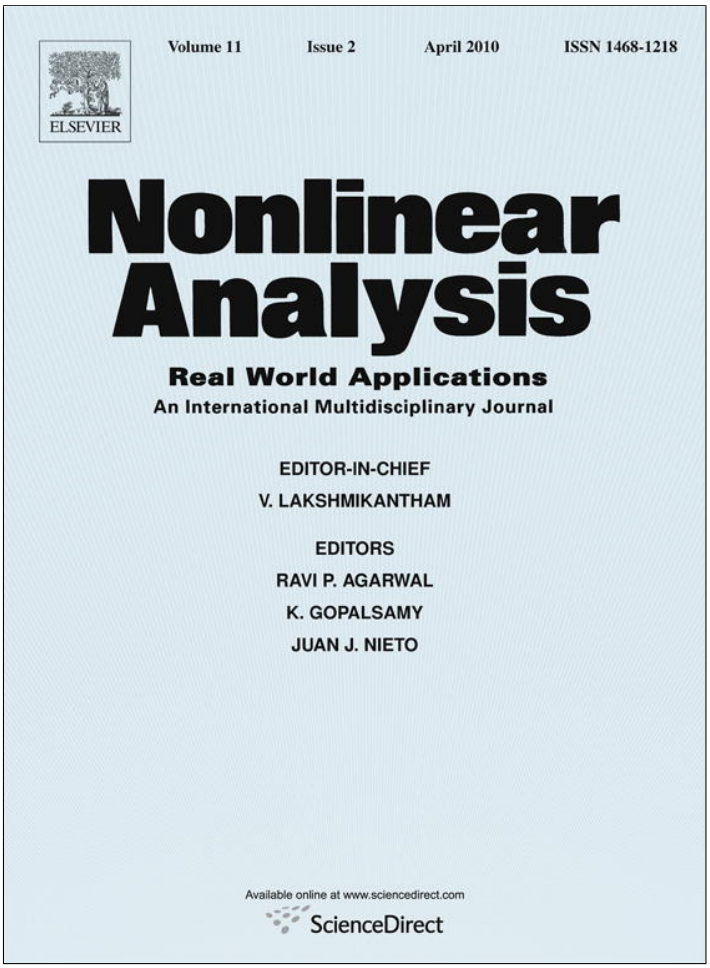

Nonlinear Analysis: Real World Applications 11 (2010) 742–749

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Evolutionary game model for a marketing cooperative with penalty for unfaithfulness

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ARTICLE INFO

Article history:
Received 16 May 2008
Accepted 12 January 2009

Keywords:
Marketing cooperative
Oligopoly
Evolutionary game dynamics

ABSTRACT

A game-theoretical model for the behaviour in a marketing cooperative is proposed. For the strategy choice an evolutionary dynamics is introduced. Considering a model with penalty for unfaithfulness and Cournot type market situation, it is shown that, if the penalty is effective then this strategy dynamics drives the players towards an attractive solution, a particular type of Nash equilibrium. A model with redistribution of penalty is also studied. For the symmetric case, on the basis of stability analysis of the strategy dynamics, in terms of the model parameters, sufficient conditions are provided for the strategy choice to converge to a strict Nash equilibrium.

1. Introduction

In agriculture we often face the situation that producers of a given product form a marketing cooperative, for the commercialization of their product. Such a cooperative in a given region may perform several activities, ranging from product processing to complex marketing, see e.g. [1]. In this paper we consider a marketing cooperative that negotiates a contracted price with large buyers, sharing risk among members of the cooperative. By the time of the actual commercialization of the product, the market price may be higher than what the cooperative can guarantee for members, negotiated on beforehand. (A normal form game model of a cooperative with a constant market price was set up in [2].) Some "unfaithful" members may be interested in selling at least a part of their product outside; the cooperative, however, can punish them for this. This conflict situation is described in terms of a game-theoretical model.

In Section 2 we set up a game model of a marketing cooperative, where every single member and the cooperative itself are the players. The strategy of a player is the proportion of the total production sold to the cooperative, while the cooperative chooses a penalty as its strategy to punish unfaithfulness. For this game an attractive solution, a particular NE (Nash equilibrium) is obtained. For the concept and the existence of an attractive solution, in a more general situation see [3,4].

Section 3 is devoted to a game model with penalty and with Cournot type oligopoly market with linear inverse demand function. For this game we also find an attractive solution.

In Section 4 an evolutionary dynamics for the strategy choice of the players is given. This dynamics (the so-called partial adaptive dynamics) is known in multispecies evolutionary game theory, see [5]. (We note that in [6], in a different situation,

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doi:10.1016/j.nonrwa.2009.01.009

another dynamic evolutionary game model was applied to the analysis of economic behaviour.) It is shown that, if the penalty is effective then this strategy dynamics drives the players towards an attractive solution.

In Section 5 a model with redistribution of penalty is studied. For the symmetric case when the members of the cooperative produce the same quantity and faithfulness is rewarded proportionally to the product sold to the cooperative, for different values of the penalty parameter, a strict NE is obtained. A stability analysis of the strategy dynamics, in terms of the model parameters, provides sufficient conditions for the strategy choice to converge to the strict NE.

2. Game model for a marketing cooperative with penalty and constant market price

Let us consider a marketing cooperative of n members producing the same product with per unit of production cost c. The cooperative guarantees to buy the whole production of its members at a contracted unit price $p > c$. However, by the time the product is available (e.g. a crop is harvested) the market may offer a better price, and a member may have a propensity to sell a part of his product on the market. Assume the total production of member i ($i \in \overline{1, n}$) in a given time period is L_i, and he sells the x_i-part ($x_i \in [0, 1]$) of his production to the cooperative at the contracted unit price p, and the $1 - x_i$-part on the market, at a unit price $q > p$. Then the profit of the ith member is

$$L_i[(p - c)x_i + (q - c)(1 - x_i)] = L_i[(p - q)x_i + q - c].$$

This attains a maximum at $x_i := 0$, risking the collapse of the cooperative.

Suppose now the cooperative decides to set a penalty for unfaithfulness, proportional to the extra profit gained from selling outside the cooperative. Then we have a conflict situation that can be modelled by the following normal form game:

the ith player is the ith member of the cooperative ($i \in \overline{1, n}$),
the $(n + 1)$th player is the cooperative,
$[0, 1]$ is the strategy set of each player,
$x \in [0, 1]^n$ is the strategy vector of the members,
$y \in [0, 1]$ is the strategy of the cooperative (penalty rate).

For any multistrategy $(x, y) \in [0, 1]^n \times [0, 1]$, the payoff (profit = revenue − cost − penalty) of the ith player is

$$\begin{aligned} f_i(x, y) &:= [(p - c)x_iL_i + (q - c)(1 - x_i)L_i - y(q - p)(1 - x_i)L_i] \\ &= L_i\{(p - c)x_i + (1 - x_i)[q - c - y(q - p)]\}, \quad (i \in \overline{1, n}) \end{aligned}$$

and for the $(n + 1)$th player the payoff is

$$f_{n+1}(x, y) := y(q - p)\sum_{j=1}^{n} L_j(1 - x_j).$$

With notation $f := (f_1, f_2, \ldots, f_{n+1})$, for the description of the cooperative we have a normal form game $([0, 1]^n \times [0, 1], f)$.

Solution of the game

In terms of vector $\mathbf{1} := (1, 1, \ldots, 1) \in [0, 1]^n$, consider the multistrategy $(\mathbf{1}, 1)$. Here the choice $x := \mathbf{1}$ means that each member sells all his production to the cooperative, $y := 1$ means that the cooperative sets the maximum penalty rate for unfaithfulness.

It is easy to see that for any $i \in \overline{1, n}$, $x \in [0, 1]^n$ and $y \in [0, 1]$ we have

$$f_i(x, 1) - f_i(\mathbf{1}, 1) = L_i\{(p - c)x_i + (1 - x_i)[q - c - (q - p)]\} - L_i(p - c) = 0, \tag{2.1}$$

and

$$f_{n+1}(\mathbf{1}, y) - f_{n+1}(\mathbf{1}, 1) = y(q - p)\sum_{j=1}^{n} L_j(1 - 1) - (q - p)\sum_{j=1}^{n} L_j(1 - 1) = 0. \tag{2.2}$$

Equality (2.1) means that, if the cooperative decides to set the maximum penalty rate $y := 1$, then the (independent) decision of any member to sell outside cannot increase his own payoff, whatever the other members do. An interpretation of (2.2) is the following: if all members sell all their production to the cooperative, the latter cannot improve its payoff deviating from the maximum penalty.

Now we will see that multistrategy $(\mathbf{1}, 1)$ turns out to be an *attractive solution* (see e.g. [3] of the game (in particular, a Nash equilibrium), in the sense of following definition. Consider an N-player game where

X_i is the strategy set of the ith player,

$X := \prod_{i=1}^{N} X_i$ the set of multistrategies,

$F_i : X \to \mathbf{R}$ the payoff of the ith player, $\quad F := (F_1, \ldots, F_N)$.

Definition 1. A multistrategy x^0 is said to be an attractive solution of the normal form game (X, F), if there exists a player $i \in \overline{1, N}$ such that the following conditions are satisfied:

$$(A) \quad F_j(x_1, \ldots, x_i^0, \ldots, x_N) \le F_j(x^0), \quad (j \in \overline{1, N} \setminus \{i\}, x_k \in X_k, k \in \overline{1, N} \setminus \{i\});$$

$$(B) \quad F_i(x_1^0, \ldots, x_i^{\cdot} \ldots, x_N^0) \le F_i(x^0), \quad (x_i \in X_i).$$

Remark 1. If in condition (A), for each $j \in \overline{1, N} \setminus \{i\}$ we choose $x_k := x_k^0 \in X_k$ for $k \in \overline{1, N} \setminus \{i, j\}$, and an arbitrary $x_j \in X_j$, we obtain that any attractive solution is a NE, too.

3. A game model with penalty and Cournot type oligopoly market

Let us consider now a generalization of the previous case, supposing that the market environment is determined by the offer of the unfaithful members of the cooperative, in terms of a Cournot type oligopoly model, and the penalty may exceed the actual extra profit of the unfaithful member.

Maintaining the notation of the previous section, let us suppose that on the outside market a Cournot type oligopoly situation with a linear inverse demand function takes place: with appropriate constants $a, b > 0$, for $x \in [0, 1]^n$ the unit price is $q(x) = a - b\sum_{i=1}^n L_i(1 - x_i)$, and $q(0) > p$ is also supposed, implying $q(x) > 0$ $(x \in [0, 1]^n)$. Notice that under this condition, also for the case of oligopoly, the members would be absolutely interested in selling outside the cooperative. (The degenerate case $q(0) = p$ will not be considered.) Now, the question again is whether a penalty is able to stabilize the cooperative.

Let $\beta \ge 1$ be a penalty parameter in the sense that on calculating the penalty, instead of y, the extra profit is multiplied by βy. Then the payoffs of the corresponding players are the following: for any multistrategy $(x, y) \in [0, 1]^n \times [0, 1]$, we have

$$\begin{aligned} f_i(x, y) &:= [(p - c)x_i L_i + (q(x) - c)(1 - x_i)L_i - \beta y(q(x) - p)(1 - x_i)L_i] \\ &= L_i\{(p - c)x_i + (1 - x_i)[q(x) - c - \beta y(q(x) - p)]\}, \quad (i \in \overline{1, n}), \end{aligned}$$

and

$$f_{n+1}(x, y) := \beta y(q(x) - p)\sum_{j=1}^n L_j(1 - x_j).$$

Now for any $i \in \overline{1, n}$, $x \in [0, 1]^n$ and $y \in [0, 1]$ we easily obtain that

$$f_i(x, 1) - f_i(\mathbf{1}, 1) = L_i(1 - \beta)(1 - x_i)(q(x) - p) \le 0,$$

and

$$f_{n+1}(\mathbf{1}, y) - f_{n+1}(\mathbf{1}, 1) = \beta y(q(x) - p)\sum_{j=1}^n L_j(1 - 1) - \beta(q(x) - p)\sum_{j=1}^n L_j(1 - 1) = 0.$$

Furthermore, in case $\beta > 1$, for $(i \in \overline{1, n})$, $x \in [0, 1]^n$ with $x_i < 1$, the inequality is strict. Indeed, the above condition $q(0) > p$ implies for all $x \in [0, 1]^n$ we have

$$q(x) - p = a - \sum_{j=1}^n L_j(1 - x_j) - p \ge a - \sum_{j=1}^n L_j - p = q(0) - p > 0.$$

Hence we obtain the following theorem.

Theorem 1. *Multistrategy* $(\mathbf{1}, 1)$ *is an* ***attractive solution*** *of the oligopoly game.*

Remark 2. Apart from being an attractive solution, for any $\beta \ge 1$ multistrategy $(\mathbf{1}, 1)$ also has the following particular property. Suppose that the cooperative sets a penalty rate different from the maximum,$y \in [0, 1[$. Then using any strategy vector $x \in [0, 1]^n$ with $x_i < 1$ for some $i \in \overline{1, n}$, we have

$$f_i(x, y) - f_i(\mathbf{1}, 1) = L_i(1 - \beta y)(1 - x_i)(q(x) - p) \begin{cases} > 0, & \text{if } \beta = 1, \\ < 0, & \text{if } \beta > 1 \text{ and } y > 1/\beta; \end{cases}$$

and

$$f_{n+1}(x, y) - f_{n+1}(x, 1) = -(1 - \beta y)(q(x) - p)\sum_{j=1}^n L_j(1 - x_j) \begin{cases} > 0, & \text{if } \beta = 1, \\ < 0, & \text{if } \beta > 1 \text{ and } y > 1/\beta. \end{cases}$$

The above inequalities can be interpreted as follows: Suppose the cooperative sets a penalty rate $y < 1$ and the ith player deviates from his equilibrium strategy (i.e. becomes unfaithful). Then, whatever the other members do, the payoff of the ith player will

(a) increase, if $\beta = 1$ and
(b) decrease, if $\beta > 1$ and $y > \frac{1}{\beta}$;
(c) whilst the payoff of the cooperative will
(d) decrease, if $\beta = 1$ and
(e) increase, if $\beta > 1$ and $y > \frac{1}{\beta}$.

4. Evolutionary dynamics for the strategy choice and its asymptotic properties

Evolutionary dynamics

We suppose the dynamic (time-dependent) strategy choice of the players can be described by an evolutionary dynamics. This is a variant of the so-called partial adaptive dynamics known in multispecies evolutionary game theory, see [5]. We note that in [6], in a different situation, another dynamic evolutionary game model has been applied to the analysis of economic behaviour. We will show that, if the penalty is effective then this strategy dynamics drives the players towards an attractive solution.

Namely, for the time-dependent strategies $x_i(t)$ and $y(t)$ let us consider the following system:

$$\dot{x}_i = x_i(1 - x_i)D_i f_i(x, y) \quad (i \in \overline{1, n}), \tag{4.3}$$

$$\dot{y} = y(1 - y)D_{n+1} f_{n+1}(x, y). \tag{4.4}$$

The intuitive background of this dynamics is the hypothesis that each player will increase (or decrease) his strategy (quantity sold to the cooperative, or penalty rate), if the corresponding strategy change results in the increase (or decrease) of his payoff; provided the rest of the players stick to their actual strategies. This means that the sign of the corresponding partial derivative will determine the process of the strategy choice.

For the *model with penalty and Cournot type oligopoly market*, let us calculate this dynamics:

For all $(x, y) \in [0, 1]^n \times [0, 1]$ we obtain

$$\dot{x}_i = x_i(1 - x_i)L_i(-1 + \beta y)[q(x) - p - bL_i(1 - x_i)] \quad (i \in \overline{1, n}), \tag{4.5}$$

$$\dot{y} = y(1 - y)\beta(q(x) - p)\sum_{j=1}^{n} L_j(1 - x_j). \tag{4.6}$$

Asymptotic properties of the strategy dynamics

Multistrategy $(\mathbf{1}, 1)$ is obviously an equilibrium for dynamics (4.5)–(4.6). For the long-term outcome of the game let us analyze the stability of this equilibrium for different values of the penalty parameter β.

Theorem 2. *In the case of* ***limited penalty*** $(\beta = 1)$ *the attractive solution* $(\mathbf{1}, 1)$ *of the "static game" is* ***unstable.*** *If the penalty is* ***effective*** $(\beta > 1)$, *any solution of the strategy dynamics starting from* $(x(0), y(0))$ *in* $]0, 1[^n \times]1/\beta, 1[$ *close enough to* $\{\mathbf{1}\} \times [1/\beta, 1]$, *tends to the attractive solution* $(\mathbf{1}, y^0)$ *with some* $y^0 \in]1/\beta, 1]$. *(The latter statement means that in the case of an effective penalty* $(\beta > 1)$, *according to the strategy dynamics, the players are driven towards an attractive solution of the game.)*

Proof. Case $\beta = 1$

Let $(x, y) \in]0, 1[^n \times]0, 1[$. Since $q(x) - p$ is greater than a fixed positive number, from (4.6) function y is strictly increasing. For the right-hand side of Eq. (4.5) we get $(-1 + y) < 0$, and near $(\mathbf{1}, 1)$, term $q(x) - p > 0$ is dominant in factor $[q(x) - p - bL_i(1 - x_i)]$, implying x_i is strictly decreasing. Therefore, the attractive solution $(\mathbf{1}, 1)$ is unstable. (Note that this is also true for the constant price case, when $b := 0$).

Case $\beta > 1$

Again, (4.6) implies y is strictly increasing. Now in Eq. (5.8), for $y > \frac{1}{\beta}$ we obtain $(-1 + \beta y) > 0$, implying x_i is strictly increasing. Hence, for every initial value $(x(0), y(0)) \in]0, 1[^n \times]1/\beta, 1[$ close enough to $\{\mathbf{1}\} \times [1/\beta, 1]$, the boundedness of the corresponding solution (x, y) implies there exist $(x^0, y^0) := \lim_\infty(x, y)$. It is easily seen that (x^0, y^0) must be an equilibrium of dynamics (4.5)–(4.6). From the above sign discussion it is clear that $x^0 = \mathbf{1}$ must hold, but for y^0 we have only $y^0 \in]1/\beta, 1]$.

Now we show that for all $y^0 \in]1/\beta, 1]$, multistrategy $(\mathbf{1}, y^0)$ is an attractive solution of the oligopoly game with penalty. Indeed, for all $i \in \overline{1, n}$, $x \in [0, 1]^n$ and $y \in [0, 1]$, an easy calculation gives

$$\begin{aligned} f_i(x, y^0) - f_i(\mathbf{1}, y^0) &= L_i(p - c)x_i + (1 - x_i)[q(x) - c - \beta y^0(q(x) - p)] - L_i(p - c) \\ &= L_i(1 - x_i)(q(x) - p)(1 - \beta y^0) < 0, \end{aligned}$$

$$f_{n+1}(\mathbf{1}, y) - f_{n+1}(\mathbf{1}, y^0) = 0,$$

implying $(\mathbf{1}, y^0)$ is an attractive solution. ■

5. Model with redistribution of penalty

In the previous theorem we have seen how an effective penalty can stabilize the cooperative. In this section we show the stabilization is even stronger, if the collected effective penalty is partly distributed among the faithful members. In particular, for the symmetric case when all members produce the same amount, sufficient conditions are obtained for the *existence* and *asymptotic stability* of a *strict* NE. Suppose a part of the total penalty collected

$$P(x, y) := \beta y[q(x) - p] \sum_{i=1}^{n} L_i(1 - x_i)$$

is redistributed amongst the members i with $x_i > 0$, according to the proportions $\alpha_i(x) \geq 0$, with

$$\alpha_i(x) = 0, \quad \text{if } x_i = 0; \qquad 0 < \sum_{i=1}^{n} \alpha_i(x) < 1 \quad \text{and} \quad D_i\alpha_i(x) > 0 \quad (i \in \overline{1, n} x \in]0, 1[^n).$$

Then the resulting payoffs are

$$f_i(x, y) := L_i[px_i + q(x)(1 - x_i) - c - \beta y(q(x) - p)(1 - x_i)] + \alpha_i(x)P(x, y) \quad (i \in \overline{1, n}) \tag{5.7}$$

and

$$f_{n+1}(x, y) := \left(1 - \sum_{i=1}^{n} \alpha_i(x)\right) \beta y(q(x) - p) \sum_{j=1}^{n} L_j(1 - x_j). \tag{5.8}$$

First we consider the problem, how to localize a NE in the case of limited penalty.

Theorem 3. *Assume $\beta = 1$. Then*

(a) *$(\mathbf{1}, y)$ is not a NE for any $y \in [0, 1]$;*
(b) *if (x, y) is a NE with $x_i < 1$ for some i, then $y = 1$.*

Proof. (a) Suppose $y \in [0, 1]$. Then for every $x \in [0, 1]^n \setminus \{\mathbf{1}\}$ and $i \in \overline{1, n}$ with $x_i < 1$ we have

$$f_i(\mathbf{1}, y) - f_i(x, y) = L_i(1 - \beta y)(p - q(x))(1 - x_i) + \alpha_i(x)[P(\mathbf{1}, y) - P(x, y)] < 0.$$

(b) Let (x, y) be a NE with $x_i < 1$ for some i. Then for all $y \in [0, 1[$ we obtain

$$f_{n+1}(x, 1) - f_{n+1}(x, y) = (1 - y)\left(1 - \sum_{i=1}^{n} \alpha_i(x)\right)(q(x) - p) \sum_{j=1}^{n} L_j(1 - x_j) > 0,$$

a contradiction with the Nash condition. ■

The rest of this section is devoted to the symmetric case when all members of the cooperative not only formally share the same strategy set (in our case $[0, 1]$), but the redistribution factors α_i have the same form, implying the payoff functions f_i have the same form, too.

5.1. Symmetric case with $\beta = 1$

Assume that all L_i's are equal, i.e., $L_i = L$ $(i \in \overline{1, n})$ for some positive L, and, $\alpha_i(x) := \frac{x_i}{n}$ for all $x \in [0, 1]^n$, $i \in \overline{1, n}$, and set $\beta := 1$. Then for all $x \in [0, 1]^n$ and $(i \in \overline{1, n})$ we have

$$f_i(x, 1) = L(p - c) + \frac{x_i}{n}\left[a - bL \sum_j (1 - x_j) - p\right] L \sum_j (1 - x_j),$$

and for the corresponding partial derivatives

$$D_i f_i(x, 1) = \frac{1}{n}\left[a - bL \sum_j (1 - x_j) - p\right] L \sum_j (1 - x_j) + \frac{x_i}{n}\left[bL^2 \sum_j (1 - x_j) - L\left(a - bL \sum_j (1 - x_j) - p\right)\right].$$

Hence we get

$$D_i f_i(\lambda \mathbf{1}, 1) = \begin{cases} > 0, & \text{if } \lambda = 0, \\ < 0, & \text{if } \lambda = 1. \end{cases}$$

Since $f_i(x, 1)$ is cubic in x_i with negative leading coefficient, there exists a unique $\lambda^* \in]0, 1[$ such that $D_i f_i(\lambda^* \mathbf{1}, 1) = 0$, and this is true for all i by symmetry.

Theorem 4. $(\lambda^*\mathbf{1}, 1)$ *is a strict NE.*

Proof. Let $\lambda^*\mathbf{1}\|x_i$ be the strategy such that player i uses x_i and all other players use λ^*. Then $f_i(\lambda^*\mathbf{1}\|0, 1) = L(p-c)$ and $f_i(\lambda^*\mathbf{1}\|1, 1) > L(p-c)$. By the above cubic argument $f_i(\lambda^*\mathbf{1}\|1, 1) > f_i(\lambda^*\mathbf{1}\|x_i, 1)$ for all $x_i \in [0, 1]$ with $\lambda^* \neq x_i$. Furthermore, since $\lambda^* \in]0, 1[$, for all $y \in [0, 1[$ we have

$$f_{n+1}(\lambda^*\mathbf{1}, 1) - f_{n+1}(\lambda^*\mathbf{1}, y) = (1-y)(1-\lambda^*)(q(x)-p)nL(1-\lambda^*) > 0.$$

Therefore, $(\lambda^*\mathbf{1}, 1)$ is a strict NE. ■

The stability analysis of this NE is considered with respect to the partial adaptive dynamics

$$\dot{x}_i = x_i(1-x_i)D_i f_i(x, y) \quad (i \in \overline{1, n}), \tag{5.9}$$
$$\dot{y} = y(1-y)D_{n+1}f_{n+1}(x, y). \tag{5.10}$$

Denote the right-hand side of (5.9) and (5.10) by Φ_i and Φ_{n+1}, respectively. By the choice of λ^*, a simple substitution shows $\Phi_{n+1}(\lambda^*\mathbf{1}, 1) = 0$ and $\Phi_i(\lambda^*\mathbf{1}, 1) = 0$ $(i \in \overline{1, n})$, i.e. $(\lambda^*\mathbf{1}, 1)$ is an equilibrium of dynamics (5.9)–(5.10). Analyzing the stability of this equilibrium by linearization, first we note that

$$D_{n+1}\Phi_{n+1}(\lambda^*\mathbf{1}, 1) = -D_{n+1}f_{n+1}(\lambda^*\mathbf{1}, 1) = -(1-\lambda^*)(q(x)-p)nL(1-\lambda^*) < 0, \tag{5.11}$$

and

$$D_i\Phi_{n+1}(\lambda^*\mathbf{1}, 1) = 0 \quad (i \in \overline{1, n}). \tag{5.12}$$

Therefore, for a stability analysis of this equilibrium it will be enough to consider the $n \times n$ Jacobian $J = [J_{ij}]$, with

$$J_{ij} := \lambda^*(1-\lambda^*)D_{ij}f_i(\lambda^*\mathbf{1}, 1).$$

An easy calculation shows that with the Kronecker δ we have

$$D_{ij}f_i(\lambda^*\mathbf{1}, 1) = -\frac{L}{n}(1+\delta_{ij})[a - 2nbL(1-\lambda^*) - p] - \frac{2\lambda^* bL^2}{n}.$$

Hence, with notation

$$H := a - 2nbL(1-\lambda^*) - p,$$
$$c := -\frac{L}{n}H - \frac{2\lambda^* bL^2}{n}, \qquad d := -\frac{L}{n}H,$$

matrix J can be written in the form $J = C + D$, where C has all entries equal to c, and D is diagonal having all diagonal entries equal to d. Thus, for the quadratic form generated by J we obtain

$$\langle x, Jx\rangle = c\left(\sum_i x_i\right)^2 + d\sum_i x_i^2.$$

Now fix a nonzero vector $v \in \mathbf{R}^n$ with $\sum_i v_i = 0$. Then $Jv = dv$. Hence we immediately obtain the following implications.

(a) $H > 0$ implies that J is negative definite,
(b) $H < 0$ implies that J has a positive eigenvalue $-\frac{L}{n}H$.

So we have proved the following theorem.

Theorem 5. (a) $H > 0$ *implies* $(\lambda^*\mathbf{1}, 1)$ *is asymptotically stable,*
(b) $H < 0$ *implies* $(\lambda^*\mathbf{1}, 1)$ *is unstable.*

Remark 3. Condition $H = a - 2nbL(1-\lambda^*) - p > 0$ means that the "oligopoly effect" is not too strong: if each member doubles his production at the equilibrium, the market price still remains above the contracted price.

5.2. Symmetric case with $\beta > 1$

In the case of an effective penalty ($\beta > 1$), depending on β, a strict NE exists:
Let us consider first

$$f_{n+1}(x, y) := \left(1 - \frac{1}{n}\sum_{i=1}^{n} x_i\right)\left[\beta y(q(x) - p)\sum_{j=1}^{n} L_j(1 - x_j)\right].$$

We see that $D_{n+1}f_{n+1}(x, y) > 0$ (unless $x_i = 1$ for all ($i \in \overline{1, n}$)). Assuming $y = 1$, we obtain

$$f_i(x, 1) = L(p - c) + \left(a - bL\sum_j (1 - x_j) - p\right)\left((1 - x_i)(1 - \beta)L + \beta L\frac{x_i}{n}\sum_j (1 - x_j)\right)$$

and

$$D_i f_i(\lambda\mathbf{1}, 1) = bL[(1 - \lambda)(1 - \beta)L + \beta L(1 - \lambda)\lambda] + (a - nbL(1 - \lambda) - p)\left(-(1 - \beta)L + \beta L(1 - \lambda) - \frac{\lambda}{n}\beta L\right).$$

Now we distinguish two cases:
Case (A) $\beta > \frac{n}{n-1}$.
We then have $f_i(\lambda\mathbf{1}\|1, 1) > f_i(\lambda\mathbf{1}\|0, 1)$ for all $\lambda \in [0, 1]$, in particular, $(0, 1)$ is not a NE.
Furthermore, a simple calculation gives $f_i(\mathbf{1}, 1) > f_i(\mathbf{1}\|x_i, 1)$ for all $x_i \in [0, 1[$; in fact, $(\mathbf{1},1)$ is the only symmetric NE.
Case (B) $\beta < \frac{n}{n-1}$
Now we additionally suppose that for the model parameters the following condition holds:

$$b < \frac{a - p}{(n + 1)L}. \tag{5.13}$$

Then by a straightforward calculation we obtain

$$D_i f_i(\lambda\mathbf{1}, 1)\begin{cases} > 0, & \text{if } \lambda = 0 \\ < 0, & \text{if } \lambda = 1. \end{cases}$$

Hence, applying again the cubic argument used in the case $\beta = 1$, there exists a unique $\lambda^* \in]0, 1[$ such that $(\lambda^*1, 1)$ is a strict NE (i.e. $f_i(\lambda^*\mathbf{1}, 1) > f_i(\lambda^*\mathbf{1}\|x_i, 1)$ for all $x_i \in [0, 1]$ with $x_i \neq \lambda^*$. In fact $(\lambda^*\mathbf{1}, 1)$ is the only symmetric NE.
Hence we have the following theorem.

Theorem 6. *(α) If $\beta > \frac{n}{n-1}$, then $(\mathbf{1}, 1)$ is the only symmetric strict NE,*
(β) If $\beta < \frac{n}{n-1}$ and condition (5.13) holds, then there exists a $\lambda^* \in]0, 1[$ such that $(\lambda^*\mathbf{1}, 1)$ is the unique strict NE.

Remark 4. Condition (5.13) can be interpreted in the following way. Either given the production of each member, the oligopoly effect in the inverse demand function is not too strong; or given the inverse demand function, the production of the members of the cooperative is not too large.

Next, for the case $1 < \beta < \frac{n}{n-1}$, under condition (5.13), we consider the problem of stability of the strict NE $(\lambda^*\mathbf{1}, 1)$ with respect to the corresponding partial adaptive dynamics. Now in system

$$\dot{x}_i = x_i(1 - x_i)D_i f_i(x, y) \quad (i \in \overline{1, n}), \tag{5.14}$$

$$\dot{y} = y(1 - y)D_{n+1}f_{n+1}(x, y) \tag{5.15}$$

the payoffs are

$$f_i(x, y) := L(p - c) + L\left[a - bL\sum_{j=1}^{n}(1 - x_j) - p\right]\left[(1 - x_i)(1 - \beta) + \beta y\frac{x_i}{n}\sum_{j=1}^{n}(1 - x_j)\right] \quad (i \in \overline{1, n}), \tag{5.16}$$

and

$$f_{n+1}(x, y) := \left(1 - \frac{1}{n}\sum_{i=1}^{n} x_i\right)\left[\beta y\left(a - bL\sum_{j=1}^{n}(1 - x_j) - p\right)\sum_{j=1}^{n} L_j(1 - x_j)\right] \quad ((x, y) \in [0, 1]^n \times [0, 1]). \tag{5.17}$$

By the choice of λ^*, $(\lambda^*\mathbf{1}, 1)$ is an equilibrium for dynamics (5.14)–(5.15). For the stability analysis of this equilibrium we apply the same method as in Section 5.1. Also in the present case each entry of the last row of the $(n + 1) \times (n + 1)$ Jacobian at $(\lambda^*\mathbf{1}, 1)$ is zero, except the last one which is negative, therefore we can again restrict ourselves to the $n \times n$ Jacobian $J = [J_{ij}]$, with

$$J_{ij} := \lambda^*(1 - \lambda^*)D_{ij}f_i(\lambda^*\mathbf{1}, 1).$$

An easy calculation shows that

$$D_{ij}f_i(\lambda^*\mathbf{1}, 1) = -\frac{L}{n}(1+\delta_{ij})[\beta(a-2nbL(1-\lambda^*)-p)+bnL(1-\beta)] - \frac{2\beta\lambda^* bL^2}{n}.$$

Defining

$$K := \beta(a-2nbL(1-\lambda^*)-p)+bnL(1-\beta)$$

and applying the same method as for $\beta = 1$, we obtain the following implications:

(i) $K > 0$ implies that J is negative definite,

(ii) $K < 0$ implies that J has a positive eigenvalue $-\frac{L}{n}K$.

From inequality (5.13) we get

$$bnL(1-\beta) > -\frac{n}{n+1}\beta(a-p),$$

implying

$$K > \beta(a-2nbL(1-\lambda^*)-p) - \frac{n}{n+1}\beta(a-p) = \frac{\beta(a-p)}{n+1} - 2\beta nbL(1-\lambda^*).$$

Note that by the choice of λ^*, for β close to $\frac{n}{n-1}$, λ^* is close to 1, implying the asymptotic stability of $(\lambda^*\mathbf{1}, 1)$. In particular, for such β condition (ii) cannot hold. Our results can be summarized in the following theorem.

Theorem 7. *Under condition (5.13) there exists $\beta_0 \in]1, n/(n-1)[$ such that for all $\beta_0 < \beta < \frac{n}{n-1}$, the NE $(\lambda^*\mathbf{1}, 1)$ in Theorem 6 is an asymptotically stable equilibrium of the strategy dynamics.*

In other words, if the oligopoly effect in the inverse demand function is not strong, the existing NE is dynamically stable against small changes.

6. Conclusion

The conflict between a marketing cooperative and its unfaithful members can be described by a multiplayer normal form game, in which the strategy of a member is the proportion of his product sold to the cooperative (the rest of his product is sold on the market). For the cooperative to set a maximum penalty rate and for the members to sell all their products to the cooperative provide an attractive solution of the game.

The partial adaptive dynamics introduced in evolutionary game theory turned out to be an appropriate tool for the description of the dynamic strategy choice of the players. Using this dynamics, a penalty for unfaithfulness has a stabilizing effect. Indeed, if the market price is determined by a Cournot type oligopoly, in the case of an effective penalty, the players are driven towards an attractive solution of the game, according to this evolutionary strategy dynamics. For the members this attractive solution is to sell all their products to the cooperative, which may guarantee the long-term survival of the cooperative.

For the symmetric case, when all members produce the same amount, the stabilization is even stronger, if the collected effective penalty is partly distributed among the faithful members. In particular, if the effective penalty is not too high and the oligopoly effect is not too strong, then a unique strict Nash equilibrium exists which is asymptotically stable for the strategy dynamics.

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